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Semiclassical Laser Theory in the Stochastic and Thermodynamic Frameworks

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The thermodynamic properties of the laser distribution in the steadily oscillating state are investigated to determine the minimum characteristic of the entropy production. First, the laser Langevin equation for five random variables is treated in the light of the stochastic calculus to deduce the photon-number rate equation $\dot{n} = -C(n - n_C) + [A/(1 + sn)](n - n_A)$, where n_C and n_A are the two constants of the fluctuation attributed to the noise forces subject to the usual fluctuation-dissipation theorem, with $n_A < 0$ for the inverted atomic population. We then combine the dynamics of the lasing mode with a model open system of the Lebowitz type with two reservoirs for which the entropy production $\sigma(p)$ is expressed and made subject to a variational principle: The modified variation scheme, the same as Prigogine's local potential method, is shown to give the exact lasing distribution p as the optimum between two distributions of thermal type with temperatures far from each other.

KEY WORDS: Stochastic calculus; information thermodynamics; reservoirs; negative temperature; entropy production; local potential.

1. INTRODUCTION

More than fifteen years has passed since the initial attempt to develop a systematic laser theory, and several almost complete quantum statistical descriptions of laser phenomena, including nonlinear fluctuation effects, are now available.[(1-3)] A mathematical analysis of the basis of such descriptions was made by Hepp and Lieb[(4)] (primarily a rigorous derivation of the laser evolution equation from the Hamiltonian dynamics supplemented by a thermodynamic-limit argument) and discussed by them further from the "exact model" point of view of the irreversible statistical mechanics of open systems.[(5,6)] The present paper aims to establish how the statistical mechanical theory can be converted into a thermodynamic description. The relevant thermodynamics is the nonequilibrium one due to Glansdorff and Prigogine,

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who developed a phenomenology to describe macroscopic structure formations in dissipative systems, their thermodynamic stability, and the nature of the associated fluctuations.[7]

As noted by Hepp,[6] the laser offers a typical example of a bifurcation phenomenon which can be described by the instability of the attractor of a given set of nonlinear deterministic, autonomous equations against external parameters. Therefore, with regard to laser theory one is interested in the nature of the predicted new equilibrium of bifurcation, often called "far from thermal equilibrium," an approach to equilibrium with the use of a probability distribution capable of describing the transient fluctuations, and then necessarily in a conceptualization, hoping for laws in accord with thermodynamics, with the *laws* in terms of the *entropy production* conforming to the Glansdorff–Prigogine context. The program was considered in fact by Graham,[8] and the present paper aims at a concrete answer within the semiclassical laser theory.

First, we invoke the revised Langevin method known as the method of stochastic differential equations (SDE), where the stochastic calculus of Itô and Stratonovich[9] provides an essential tool. We also need a framework in which to combine thermodynamics and information theory—*information thermodynamics* according to Ingarden and Kossakowski[10]—which we will discuss here only to the extent to which it is needed to formulate the entropy production relevant to the laser SDE. Then we proceed to the physics of lasers, including the Dicke superradiance,[11] and obtain the minimum property of a quantity associated with the entropy production in the steady state and its possible extension to transient states. In this paper, we concentrate on the well-explored laser distribution in the steady oscillation. The transient processes will be treated in a subsequent paper.[31]

In their exposition of quantum theory of an optical maser, Scully and Lamb[12] discussed the specific nature of the problem, emphasizing that the lasing state of the electromagnetic field system is in contact with a steady but nonthermal reservoir capable of supplying energy to be converted into a nearly monochromatic oscillation at the optical frequency outside the cavity: Its thermodynamic openness should therefore be characterized by the presence of energy flows between the system and, actually, two reservoirs of thermal type, viz. the reservoirs of the energy supply and the receiver. The entropy production for such an open system in contact with more than two thermal reservoirs has been discussed by Lebowitz *et al.*,[13–15] who proposed a formula of the entropy production

$$\sigma(\rho) = (d/dt)H(\rho) - k_B \sum_i \beta_i J_i$$

where J_i is the energy flow from the ith reservoir to the system, $k_B\beta_i$ is the inverse temperature of the ith reservoir, and $H(\rho)$ is the Boltzmann H-function of the density matrix ρ (probability density p in the classical sense) for the system to obey a relevant evolution equation:

$$H(\rho) = -k_B \operatorname{Tr}(\rho \log \rho), \qquad (d/dt)\rho = L\rho$$

We use the above formula, showing its consistency with the information thermodynamic argument and leaving the examination of its validity to a later time.

All we demonstrate here is that the laser is an ideal example of the Lebowitz type of open system: The stochastic calculus of Itô and Stratonovich provides us with an adequate tool to find the thermodynamic characterization of the reservoirs, i.e., to determine the β_i and J_i in the formula for $\sigma(p)$ for the laser system, in particular revealing the effective *negative temperature* in the lasing, the source of the immense power that is emitted. We remark also that because of this large difference in temperature between the different reservoirs the method of expanding the $\sigma(p)$ in terms of the temperature difference[13] is inadequate, so that a consistent variation scheme is necessary, in accordance with our previous formulation of variation,[16] which revises Prigogine's *local potential* concept.[7,27,28] In this way, we reach the main conclusion of the present paper: The steady-lasing distribution p for the power of the light being emitted is derived from a modified variational principle of the local-potential type

$$\sigma(\hat{p}, p) = \text{minimum with respect to } \hat{p}, \quad \hat{p}_{\min} = p$$

where $\sigma(\hat{p}, p)$ satisfies $\sigma(\hat{p} = p, p) = \sigma(p)$. Then, the algebraic sum of the energy flows $J_{\text{atom}} + J_{\text{field}}$ vanishes, while the corresponding thermodynamic entropy production $-\beta_{\text{atom}} J_{\text{atom}} - \beta_{\text{cavity}} J_{\text{cavity}}$, equal to the minimized value $\sigma(p)$, does not vanish. This nonvanishing of the steady entropy production deduced from the minimization reflects the original Prigogine thermodynamic context, i.e., the entropy production minimum theorem in the presence of a force constraint.[17] Finally, the resulting extensive flow, $J_{\text{atom}} > 0$, expressed in terms of the controllable force $|\beta_{\text{atom}}|$ represents the Onsager relation in such a far-from-equilibrium situation.

2. LASER STOCHASTIC DIFFERENTIAL EQUATION

A laser system is composed of the following three dynamical constituents: (1) lasing modes of the electromagnetic field in a cavity; (2) a number of atoms active in the electromagnetic field in the cavity; (3) heat reservoirs with which the above two species are in contact.

The effects of the reservoirs upon the systematic motion of the other subsystems may be given in a stochastic description: For the simplest case (the single-mode, two-level-atom theory), therefore, the following set of SDE for five random variables b, b^*, R, R^*, and Z contains the essence of the laser action:

$$\begin{aligned} db &= -[(\kappa + i\omega)b + (i\lambda/\sqrt{N})R]\,dt + dw \\ db^* &= -[(\kappa - i\omega)b^* + (-i\lambda/\sqrt{N})R^*]\,dt + dw^* \\ dR &= -[(\gamma + i\omega_0)R - (i2\lambda/\sqrt{N})Zb]\,dt + dW \\ dR^* &= -[(\gamma - i\omega_0)R^* + (i2\lambda/\sqrt{N})Zb^*]\,dt + dW^* \\ dZ &= [-\gamma(Z - Z_0) + (i\lambda/\sqrt{N})(b^*R - bR^*)]\,dt + dW_z \end{aligned} \tag{1}$$

Here, the notation is mostly standard[1-3]: κ, $\gamma_\perp$, and $\gamma_\parallel$ are the three damping constants for the complex amplitude of the mode b, the complex atomic dipole R, and the inversion of the atomic population Z $[=\frac{1}{2}(N_{\text{upper}} - N_{\text{lower}})]$, respectively. The parameter of coupling between the dipole and the mode is denoted by λ divided by $\sqrt{N}$, where N is the total number of atoms (this follows Hepp and Lieb[4]). The capital letters R and Z are used for the atomic variables to indicate that these are *extensive variables*, i.e., variables that can take on values of the order of N, which we express as

$$R, R^*, Z = O(N) \tag{2}$$

Hepp and Lieb have shown that such a semiphenomenological set of equations can be deduced rigorously from the Hamiltonian dynamics containing the reservoir freedom under suitable assumptions, especially the assumption of a specific model for the reservoir, called the "singular reservoir," which has the effect of eliminating memory terms and, in the limit of $N \to \infty$, closes the equations in terms of only the three collective modes of the atoms together with the Gaussian noise terms (dw, dW, their complex conjugates, and dW_z). They also discussed the extensivity feature of all the variables including the noise, showing that

$$b, b^* = O(N^{1/2}); \qquad w, w^* = O(1); \qquad W, W^*, W_z = O(N^{1/2}) \tag{3}$$

so that Eq. (1) can be normalized to the form

$$dx_\mu = h_\mu(x)\,dt + (1/\sqrt{N})\,dw_\mu \tag{4}$$

where x_μ, $h_\mu(x)$, and w_μ are all intensive quantities. This is the standard form of SDE involving an extensiveness parameter Ω $(=N)$ in the sense of van Kampen,[18] where the proportionality to $N^{-1/2}$ of the noise terms may be regarded as a guarantee that the actual stochastic process can be approximated by a diffusion process.[19]

A basic question about the use of SDE (1) for the laser problem is whether it is an adequate substitute for the more rigorous quantum master equation, and the answer to this question, according to Hepp and Lieb, is that in the thermodynamic limit the result obtained from the leading terms of the SDE without the noise (i.e., the classical, mean field theory) should agree with what can be expected from the quantum dynamical theory in the same order. In the Appendix we compare the reduced Fokker–Planck equations from the present SDE with the system-size expansion[18] of the laser and superradiance master equations to examine the next order and to show the extent of the adequacy of (1) with a suitable inclusion of the quantum fluctuation effects.

Here, we note the physical result of the leading deterministic equation:

$$\dot{\tilde{b}} = -(\kappa + i\,\Delta\omega)\tilde{b} + (i\lambda/\sqrt{N})\tilde{R}, \qquad \dot{\tilde{b}}^* = \text{c.c.} \quad (\Delta\omega = \omega - \Omega) \tag{5}$$

$$\dot{\tilde{R}} = -(\gamma_\perp + i\,\Delta\omega_0)\tilde{R} - (2i\lambda/\sqrt{N})Z\tilde{b}, \quad \dot{\tilde{R}}^* = \text{c.c.} \quad (\Delta\omega_0 = \omega_0 - \Omega) \tag{6}$$

$$\dot{Z} = -\gamma_\parallel(Z - Z_0) + (i\lambda/\sqrt{N})(\tilde{b}^*\tilde{R} - \tilde{b}\tilde{R}^*) \tag{7}$$

in the rotating coordinate system with frequency Ω ($\tilde{b} = be^{-i\Omega t}$, etc.). The condition of the vanishing of the left-hand sides of (5)–(7) yields the laser bifurcation:

Trivial attractor:

$$b = R = 0$$

$$Z = Z_0 < Z_{\text{th}} \equiv \frac{N\kappa\gamma_\perp}{2\lambda^2}\left[1 + \left(\frac{\omega - \omega_0}{\kappa + \gamma_\perp}\right)^2\right] \tag{8}$$

Nontrivial attractor:

$$\Omega = \frac{\gamma_\perp\omega + \kappa\omega_0}{\gamma_\perp + \kappa}, \qquad Z = Z_{\text{th}} < Z_0$$

$$n \equiv b^*b = \frac{N\gamma_\perp\gamma_\parallel}{4\lambda^2}\left[1 + \left(\frac{\omega - \omega_0}{\gamma_\perp}\right)^2\right]\frac{Z_0 - Z_{\text{th}}}{Z_{\text{th}}} \tag{9}$$

The latter situation corresponds to the lasing state. An approach to the lasing state may be described by an *adiabatic reduction* of the atomic dipole variable $\tilde{R}$

$$\tilde{R} = i\frac{2\lambda Z}{\sqrt{N}\,\gamma_\perp}\,\tilde{b}$$

from $\dot{\tilde{R}} = 0$ in (6), which is substituted into Eqs. (5) and (7), leading to

$$\begin{aligned} \dot{n} &= -2\kappa n + \frac{4\lambda^2}{N\gamma_\perp}\frac{nZ}{1 + (\omega - \omega_0)^2/{\gamma_\perp}^2} \\ \dot{Z} &= -\gamma_\parallel(Z - Z_0) - \frac{4\lambda^2}{N\gamma_\perp}\frac{nZ}{1 + (\omega - \omega_0)^2/{\gamma_\perp}^2} \end{aligned} \tag{10}$$

This is the Statz–de Mars equation.[20] The growth of the number of photons n from almost zero values to the saturation value $\sim 10^{12}$ given by (9) may be studied by an equation of the type

$$\dot{n} = -2\kappa n + \frac{(4\lambda^2/N\gamma_\perp)Z_0 n}{1 + (\omega - \omega_0)^2/\gamma_\perp{}^2 + (4\lambda^2/N\gamma_\parallel\gamma_\perp)n} \tag{11}$$

which is obtained from the further adiabatic reduction of Z through $\dot{Z} = 0$.

3. ELEMENTS OF STOCHASTIC CALCULUS APPLIED TO THE LASER SDE

We are led by the manipulation just mentioned in the last section to the hope that the adiabatic reduction method can be extended to the SDE, i.e., to an inclusion of the noise terms. This requires a careful analysis, because the Gaussian noise terms, when interpreted as the white noise, are such that they have no differential almost all the time. One finds, then, a guide from the stochastic calculus of Itô,[9] which utilizes the *symmetric stochastic integral of Stratonovich*.[21] We first summarize its elements as follows: A symmetric multiplication of the differential of a stochastic variable X by another variable Y denoted by $Y \circ dX$ is defined from the corresponding stochastic integral in an interval I

$$Y \circ dX(I) = \text{lip} \sum_{i=1}^{n} \tfrac{1}{2}[Y(t_{i-1}) + Y(t_i)]\, dX(I_i)$$

(lip denotes the *limit in the probability* when the maximum length of the subintervals tends to zero), which is related to the usual Itô integral as

$$\begin{aligned} &\text{lip} \sum_{i=i}^{n} Y(t_i)\, dX(I_i) + \tfrac{1}{2} \sum_{i=i}^{n} dY(I_i)\, dX(I_i) \\ &\quad = Y\, dX(I) + \tfrac{1}{2}(dY\, dX)(I) \end{aligned}$$

The basic rules in the use of such symmetric products are:

(i) $Y \circ dX = Y\, dX + \frac{1}{2}\, dX\, dY$.
(ii) $Y \circ (Z \circ dX) = (YZ) \circ dX$.
(iii) $df(X_1, X_2, \ldots, X_d) = \sum_{\mu=1}^{d} [\partial f(X_1 \cdots X_d)/\partial X_\mu] \circ dX_\mu$.

Remark. In (i), $Y \circ dX = Y\, dX$, if X or Y is a function of t of the bounded variation, so that in particular $f(X) \circ dt = f(X)\, dt$.

Roughly speaking, therefore, the differential calculus with the symmetric multiplication may be utilized just as in the usual calculus, which suggests that the adiabatic reduction method can be made equally applicable to the SDE first by rearranging it in the Stratonovich sense and then by using the basic rules (i)–(iii). This ansatz should be established mathematically, but here we go on further by assuming its validity. A plausibility argument is available from the consideration that, once the Gaussian white noise in the SDE is regarded as the limit of a series of real noise (the so-called "colored noise"), then a theorem by Wong and Zakai[(22)] ensures that the limit of the corresponding solutions of the SDE are identified with that of the limiting SDE in the Stratonovich sense.[2] Before exhibiting our results for the adiabatic reduction, we must first fix the physical property of the white noise set up in the starting SDE (1), which will be given in the following discussion of the limit $\lambda \to 0$.

In the vanishing coupling between the mode and the atoms, $\lambda = 0$, the laser SDE (1) reduces to two groups of linear Brownian motions; viz. the Brownian motion of photons and that of atoms, the latter being considered as the Brownian motion of spins[(23)]:

Brownian motion of photons:

$$db = -(\kappa + i\omega)b\, dt + dw, \qquad db^* = -(\kappa - i\omega)b^*\, dt + dw^* \tag{12}$$

$$\langle dw^*(t)\, dw(t)\rangle = 2\kappa n\, dt, \qquad \langle dw(t)\, dw(t)\rangle = 0 \tag{13}$$

where the familiar Einstein relation yields

$$\bar{n} = \langle b^* b\rangle_{\text{eq}} = O(1) \tag{14}$$

We now show that the stochastic calculus can be used for the transformation of SDE (12) into another SDE in the intensity–phase representation:

$$b = \sqrt{n}\, e^{i\varphi}, \quad b^* = \text{c.c.} \qquad \text{or} \qquad n = b^* b, \quad \varphi = \frac{1}{2i} \log \frac{b}{b^*}$$

The basic rules (i)–(iii) together with SDE (12) enable us to obtain

$$\begin{aligned} dn &= b^* \circ db + b \circ db^* = -2\kappa n\, dt + (b^* \circ dw + b \circ dw^*) \\ &= -2\kappa n\, dt + (b^*\, dw + b\, dw^*) + \tfrac{1}{2}(db^*\, dw + db\, dw^*) \end{aligned}$$

the last term being replaced by $\frac{1}{2} \times 4\kappa\bar{n}\, dt$ in accordance with the Einstein relation (13), so that (after a similar manipulation for $d\varphi$)

$$\begin{aligned} dn &= -2\kappa(n - \bar{n})\, dt + (b^*\, dw + b\, dw^*) \\ d\varphi &= -\omega\, dt + \frac{1}{2i}\left(\frac{1}{b}\, dw - \frac{1}{b^*}\, dw^*\right) \end{aligned} \tag{15}$$

and

$$\langle dn\, dn\rangle = 4\kappa\bar{n}n\, dt, \qquad \langle d\varphi\, d\varphi\rangle = (\kappa\bar{n}/n)\, dt, \qquad \langle dn\, d\varphi\rangle = 0 \tag{16}$$

[2] We are indebted to Prof. L. Arnold and Dr. W. Horsthemke for calling our attention to Wong and Zakai's theorem.

It is then easy to write the associated diffusion equation, or the Fokker–Planck (FP) equation, according to the established rule[9,21]:

$$\text{(FP)} \qquad \frac{\partial p}{\partial t} = 2\kappa \frac{\partial}{\partial n}(n - \bar{n})p + 2\kappa\bar{n}\frac{\partial^2}{\partial n^2} np + \omega\frac{\partial p}{\partial \varphi} + \frac{\kappa\bar{n}}{2n}\frac{\partial^2 p}{\partial \varphi^2}$$

$$= 2\kappa\frac{\partial}{\partial n}\left(p + \bar{n}\frac{\partial p}{\partial n}\right) + \left(\omega\frac{\partial p}{\partial \varphi} + \frac{\kappa\bar{n}}{2n}\frac{\partial^2 p}{\partial \varphi^2}\right) \tag{17}$$

In particular,

$$p_{\text{st}} + \bar{n}\,\partial p_{\text{st}}/\partial n = 0 \tag{18}$$

For the steady-state distribution

$$p_{\text{st}}(n) = \text{const}\cdot\exp(-n/\bar{n}) \tag{19}$$

Note that the intensity n (photon number) has the expected limit $\bar{n}$ in Eq. (15), which has been deduced from the symmetric product $b^* \circ dw + b \circ dw^*$ by rewriting it in terms of the Itô integral plus the correction according to the rule (i). Note that the Itô integral is characterized by its vanishing expectation, called the *martingale* property.

Brownian motion of spins:

$$\begin{gathered} dR = -(\gamma_\perp + i\omega_0)R\,dt + dW, \qquad dR^* = -(\gamma_\perp - i\omega_0)R^*\,dt + dW^* \\ dZ = -\gamma_\parallel(Z - Z_0)\,dt + dW_z \end{gathered} \tag{20}$$

The equations for dR and dR^* are of the same structure as the Brownian motion of photons, so that we only need to specify the variance parameter. Similarly, the last equation is also a typical linear process (with a nonzero equilibrium value) characterized by another variance parameter, the whole thus representing a Bloch-type relaxation process. An isotropic distribution about the mean value will be assumed in the equilibrium state, for simplicity, which can be expressed as $\langle dW^*\,dW\rangle = 2\gamma_\perp\bar{M}\,dt$, $\langle dW_z\,dW_z\rangle = \gamma_\parallel\bar{M}\,dt$, all others vanish, where

$$\bar{M} = \langle R^*R\rangle_{\text{eq}} = 2\langle(Z - Z_0)^2\rangle_{\text{eq}} = O(N) \tag{21}$$

We will show that the single variance parameter $\bar{M}$ can be determined from a detailed-balance argument in the presence of coupling between the mode and the atomic dipoles in equilibrium and ensure that $\bar{M} = O(N)$.

4. ADIABATIC REDUCTION AND THE REDUCED PROCESSES

4.1. The Dicke Superradiance[11,24]: $\gamma_\perp = \gamma_\parallel = 0$ in (1)

This is the case in which the atomic subsystem is switched off from the contact with its own reservoir. Then, the whole process becomes analogous to the classical Ornstein–Uhlenbeck process, where the adiabatic reduction of the velocity process leading to induced diffusion in real space[25] corresponds to the present adiabatic reduction of photons and the resulting induced diffusion of spins. The term "adiabatic" implies that the motion of the former is so rapid that its action on the latter can be regarded as exerted through every instantaneous value of the former, its time derivative being ignored. Thus, in the resonance condition $\omega = \omega_0$ $(=\Omega)$ for simplicity in Eqs. (5)–(7), the reduction can be made from

$$b \circ dt = \frac{-i\lambda}{\kappa\sqrt{N}} R \circ dt + \frac{dw}{\kappa}, \qquad b^* \circ dt = \text{c.c.} \tag{22}$$

which are substituted into the rest of the SDE:

$$\begin{aligned} dR &= \frac{2\lambda^2}{\kappa N} ZR \circ dt + \frac{2i\lambda}{\kappa\sqrt{N}} Z \circ dw, \qquad dR^* = \text{c.c.} \\ dZ &= -\frac{2\lambda^2}{\kappa N} RR^* \circ dt + \frac{-i\lambda}{\kappa\sqrt{N}} (R^* \circ dw - R \circ dw^*) \end{aligned} \tag{23}$$

Here, the Stratonovich symmetric product is important, because only by this interpretation in (22) is the radius of the "spin" $\|R\|$, $\|R\|^2 = R^*R + Z^2$ $[=(N+1)/2$ in the classical sense, see Appendix, (A17)] conserved, i.e.,

$$d\|R\|^2 = R^* \circ dR + R \circ dR^* + 2Z \circ dZ = 0$$

so that the resulting diffusion is restricted to a sphere.[3] This can be most clearly represented by a Landau–Lifschitz type of damping with noise

$$dr = I(\hat{z} \times r) \times r \circ dt + 2\|R\|^{-1/2} r \times \circ d\bar{w} \tag{24}$$

for the intensive vectors

$$r = \|R\|^{-1}(\text{Re R, Im R}, Z) \qquad \bar{w} = \lambda \|R\|^{1/2} \kappa N^{1/2} (\text{Re } w, \text{Im } w, 0) \tag{25}$$

and the time-scaling factor

$$I = 2\lambda^2 \|R\| / \kappa N \tag{26}$$

Again, the use of the stochastic calculus provides the spherical coordinate representation $r = \sin\theta\, e^{i\varphi}$, $r^* = \text{c.c.}$, $z = \cos\theta$ as follows:

$$\begin{aligned} d\theta &= \sin\theta\, I\, dt + i\|R\|^{-1/2}(e^{-i\varphi} \circ d\bar{w} - e^{i\varphi} \circ d\bar{w}^*) \\ d\varphi &= \|R\|^{-1/2} \cot\theta \circ (e^{-i\varphi} \circ d\bar{w} + e^{i\varphi} \circ d\bar{w}^*) \end{aligned} \tag{27}$$

$$\langle d\theta\, d\theta \rangle = \frac{2\bar{n}}{\|R\|} I\, dt, \qquad \langle d\varphi\, d\varphi \rangle = \frac{2\bar{n}}{\|R\|} \cot^2\theta\, I\, dt, \qquad \langle d\theta\, d\varphi \rangle = 0 \tag{28}$$

[3] Note that the radius $\|R\|$ is not conserved, e.g., $\|R\| \propto e(4\lambda^2/\kappa N)\bar{n}t$, if the right-hand side of (22) is interpreted as the Itô integral.

The deterministic equation for the inversion is given by

$$\dot{Z} = (-2\lambda^2/\kappa N)(\|R\|^2 - Z^2 + 2\bar{n}Z) \tag{29}$$

4.2. The Ordinary Laser: $0 \neq \varkappa \ll \gamma_\perp, \gamma_\parallel$ in (1)

The adiabatic reduction here implies that the motion of atomic dipoles is rapid, so that as a converse to (21) we have (again, resonance $\omega = \omega_0$ is assumed)

$$R \circ dt = \frac{i2\lambda}{\gamma_\perp\sqrt{N}} Zb \circ dt + \frac{dW}{\gamma_\perp}, \qquad R^* \circ dt = \text{c.c.} \tag{30}$$

which are substituted into the rest of the SDE (1):

$$db = \left(-\kappa b + \frac{2\lambda^2}{\gamma_\perp N} Zb\right) \circ dt + \frac{-i\lambda}{\gamma_\perp N} dW + dw, \qquad db^* = \text{c.c.} \tag{31}$$

$$dZ = \left[-\gamma_\parallel(Z - Z_0) - \frac{4\lambda^2}{\gamma_\parallel\gamma_\perp N} nZ\right] \circ dt + \frac{i\lambda}{\gamma_\perp\sqrt{N}}(b^* \circ dW - b \circ dW^*) \tag{32}$$

Let us introduce the saturation factor s defined by

$$s = 4\lambda^2/\gamma_\parallel\gamma_\perp N = O(N^{-1}). \tag{33}$$

This together with the transformation into the n–φ representation yields

$$\begin{aligned} dn &= (-2\kappa n + \gamma_\parallel snZ)\, dt + (b^* \circ dw + b \circ dw^*) \\ &\quad + \frac{-i\lambda}{\gamma_\perp\sqrt{N}}(b^* \circ dW - b \circ dW^*) \end{aligned} \tag{34}$$

$$d\varphi = \frac{1}{2i}\left(\frac{1}{b} \circ dw - \frac{1}{b^*} \circ dw^*\right) + \frac{\lambda}{2\gamma_\perp\sqrt{N}}\left(\frac{1}{b} \circ dW - \frac{1}{b^*} \circ dW^*\right) \tag{35}$$

$$dZ = -\gamma_\parallel[(1 + sn)Z - Z_0]\, dt + \frac{i\lambda}{\gamma_\perp\sqrt{N}}(b^* \circ dW - b \circ dW^*) \tag{36}$$

We now come to one of the important points of the manipulation, namely the corrections in $O(N^{-1})$ to the leading deterministic equations for n and Z [i.e., the Statz–de Mars equation (10)]:

$$\dot{n} = -2\kappa(n - \bar{n}) + \gamma_\parallel s(nZ + \tfrac{1}{2}\overline{M}) \tag{37}$$

$$\dot{Z} = -\gamma_\parallel(Z - Z_0) - \gamma_\parallel s(nZ + \tfrac{1}{2}\overline{M}) \tag{38}$$

As we have seen already, the correction term proportional to $\bar{n}$ arises from the symmetric product $(b^* \circ dw + b \circ dw^*)$. Here, it can be observed that the correction proportional to $\overline{M}$ arises from a similar product involving dW

and dW^* in (34) and (36) as a consequence of the adiabatic reduction with our ansatz, the physical significance of which we now show.

At thermal equilibrium (no pumping of the atomic population) the left-hand sides of the deterministic equations (37) and (38) must vanish automatically, which implies that $n = \bar{n}$ and $Z = Z_0$, and hence

$$\bar{M} = -2Z_0\bar{n} = (\bar{N}_{\text{low}} - \bar{N}_{\text{up}})\bar{n} \tag{39}$$

Thus the ansatz of the adiabatic reduction with the symmetric multiplication rule has provided a condition of detailed balance which determines $\bar{M}$. If we use the usual Boltzmann factor for $\bar{N}_{\text{low}} - \bar{N}_{\text{up}}$ and the Planck formula with the correction for spontaneous emission for $\bar{n}$, we get the simple result

$$\bar{M} = \tfrac{1}{2}N \tag{40}$$

which is the best result consistent with the quantum statistics (see the Appendix). We now also argue that the same correction provides an effective photon number associated with the pumped atomic reservoir. Let us perform one more adiabatic reduction by assuming $dZ = 0$ in (36) and substitute the expression

$$\begin{aligned} Z \circ dt = {} & \frac{Z_0}{1 + sn} \circ dt + \frac{i\lambda}{\gamma_{\parallel}\gamma_{\perp}\sqrt{N}} \frac{1}{1 + sn} \circ (b^* \circ dW - b \circ dW^*) \\ & + \frac{1}{\gamma_{\parallel}} \frac{1}{1 + sn} \circ dW_z \end{aligned} \tag{41}$$

into (34), so that

$$\begin{aligned} dn = {} & \left(-2\kappa + \frac{\gamma_{\parallel} s Z_0}{1 + sn}\right) n \circ dt + \Bigg\{ b^* \circ dw + b \circ dw^* \\ & + \frac{-i\lambda}{\gamma_{\perp}\sqrt{N}} \left(\frac{b^*}{1 + sn} \circ dW - \frac{b}{1 + sn} \circ dW^*\right) + \frac{sn}{1 + sn} \circ dW_z \Bigg\} \end{aligned} \tag{42}$$

and

$$\langle dn\, dn \rangle = \left(4\kappa\bar{n}n + \frac{\gamma_{\parallel}\bar{M}sn}{1 + sn}\right) dt \tag{42a}$$

This gives a deterministic equation for the single variable n

$$\dot{n} = -2\kappa(n - \bar{n}) + \gamma_{\parallel} s \frac{Z_0 n + \frac{1}{2}\bar{M}}{1 + sn} + O(n^2 N^{-2}) \tag{43}$$

The same equation without the last correction term may be obtained by the adiabatic reduction of Z in Eqs. (37) and (38). Let us rewrite this equation by introducing

$$n_C = \bar{n}, \qquad n_A = -(\bar{M}/2Z_0) \quad [= O(1)] \tag{44}$$

that is

$$\dot{n} = -2\kappa(n - n_C) - \frac{\gamma_{\|} s(-Z_0)}{1 + sn}(n - n_A) \tag{45}$$

Then, a clear physical meaning can be assigned to the right-hand side of this equation: The time rate of change of the photon number consists of two parts: the linear decay to its thermal value n_C in contact with the cavity (the cavity loss), and a nonlinear decay to the value n_A in contact with the atomic reservoir (the power loss) provided that $n_A > 0$. The latter value defined by (44) has a real meaning as long as $Z_0 < 0$, i.e., the atomic population is ordinary. Once the population is inverted so that $Z_0 > 0$, however, Eq. (45) must be rewritten as

$$\dot{n} = -2\kappa(n - n_C) + \frac{\gamma_{\|} s Z_0}{1 + sn}(n + |n_A|) \tag{46}$$

in which the correction term n_A is difficult to understand. But we assert that even for such a situation the negative value n_A has a physical significance from a thermodynamic point of view; it represents *the effective negative temperature of the atomic reservoir*, just as n_C represents the temperature of the cavity. This will be formulated in the next section.

5. ENTROPY PRODUCTION ASSOCIATED WITH THE LASER SDE

The concept of entropy production, extended from the pure thermodynamic regime to that for open systems in which thermal reservoirs are described as equilibrium, inexhaustible degrees of freedom and a dynamical system is introduced in a statistical mechanical framework, has been discussed by Bergmann and Lebowitz,[13] Lebowitz,[14] and also recently by Spohn and Lebowitz[15] (see also Hepp[6]). They provided a simple expression for the entropy production σ which reduces to the time derivative of the relative entropy

$$S(\rho|\rho_\beta) = \mathrm{Tr}\, \rho(-\log \rho + \log \rho_\beta) \quad \text{(in the quantum sense)} \tag{47}$$

for an open system of the following type: A dynamical system represented by the distribution ρ is in contact with one thermal reservoir of temperature β^{-1} and its evolution obeys

$$d\rho/dt = L\rho, \qquad L\rho_\beta = 0 \qquad \text{with} \quad \rho_\beta \propto e^{-\beta E} \tag{48}$$

for which

$$\sigma(\rho) = -\mathrm{Tr}[(L\rho) \log \rho] - \beta\, \mathrm{Tr}[(L\rho)E] \tag{49}$$

(the convention that the Boltzmann constant is one is used). The interpretation of the right-hand side of (49) is that the first term represents the rate of entropy for the dynamical system—the information entropy in terms of ρ—and the second term that for the reservoir, which yields the well-known thermodynamic description:

$$-\beta\,\mathrm{Tr}[(L\rho)E] = -\beta\,\mathrm{Tr}[\rho(L^*E)] = -\beta J = \beta\, dQ/dt \tag{50}$$

It implies that the flow of energy into the dynamical system J (calculated by the time derivative of the system Hamiltonian) is just equal to the negative heat flow into the reservoir in the absence of external work, conforming to the first law of thermodynamics. Therefore,

$$\sigma(\rho) = \frac{dS}{dt} + \beta\frac{dQ}{dt} \geqslant 0 \tag{51}$$

where the inequality, due to the dissipative nature of the evolution operator L, expresses the second law of thermodynamics.

On the above basis the expression for σ can be reasonably extended to those open systems in which the dynamical system is in contact with several thermal reservoirs as follows:

$$\begin{aligned}\sigma(\rho) &= \frac{dS}{dt} - \sum_i \beta_i J_i \\ &= -\mathrm{Tr}\left[\sum_i (L_i\rho)\log\rho\right] - \sum_i \beta_i\,\mathrm{Tr}[(L_i\rho)E] \\ &= \sum_i \sigma_i(\rho)\end{aligned} \tag{52}$$

where β_i^{-1} is the temperature of the ith reservoir and L_i the evolution operator for the dynamical system when its contact with every reservoir except for the ith one is switched off. The dynamical system ρ is assumed to evolve by the total evolution operator $L = \sum_i L_i$, whereas the heat exchange between the system and the reservoirs is supposed to take place in such a way that the associated entropy change is just equal to the sum of the individual entropy changes. The formula (52) has not been fully justified, its validity depending possibly on the weak coupling treatment of the system–reservoir interaction. However, we show that our laser system treated by means of the previous adiabatic reduction method of SDE is completely adapted to the formula, which enables us to put the whole analysis into the thermodynamic framework.

Let us first summarize the information-thermodynamic entropy production associated with a SDE: $dx_\mu = h_\mu(x)\,dt + \sigma_{\mu i}(x)\circ dw_i$, where the $w_i(t)$

are the standard Brownian motion $\langle dw_i\, dw_j\rangle = \delta_{ij}$. The Fokker–Planck equation which describes the probability of the process $x_\mu(t)$ is

$$\begin{aligned}\frac{\partial p}{\partial t} &= -\frac{\partial}{\partial x_\mu}(h_\mu p) + \frac{1}{2}\frac{\partial}{\partial x_\mu}\left(\sigma_{\mu i}\frac{\partial}{\partial x_\nu}\sigma_{\nu i}p\right)\\ &= -\frac{\partial}{\partial x_\mu}(b_\mu p) + \frac{1}{2}\frac{\partial}{\partial x_\mu}\left(g_{\mu\nu}\frac{\partial p}{\partial x_\nu}\right), \qquad g_{\mu\nu} = \sigma_{\mu i}\sigma_{\nu i}\\ &= \frac{1}{2}\frac{\partial}{\partial x_\mu} g_{\mu\nu} p\left(\beta\frac{\partial E}{\partial x_\nu} + \frac{\partial}{\partial x_\nu}\log p\right) = Lp \end{aligned} \tag{53}$$

The last expression is due to the assumption of a single reservoir with temperature β^{-1} for the process which satisfies the condition (48) (it imposes a potential condition on the drift vector b_μ as a generalization of the Einstein relation). If we use the notation of the scalar product $\langle p, X\rangle$ for the average of X over the probability density p instead of the quantum trace operation, then

$$\begin{aligned}-\langle Lp, \log p\rangle &= \frac{1}{2}\left\langle p, g_{\mu\nu}\left(\frac{\partial \log p}{\partial x_\mu} + \beta\frac{\partial E}{\partial x_\mu}\right)\frac{\partial \log p}{\partial x_\nu}\right\rangle\\ -\beta\langle p, L^*E\rangle &= -\frac{\beta}{2}\left\langle p, g_{\mu\nu}\frac{\partial \log p}{\partial x_\mu}\frac{\partial E}{\partial x_\nu}\right\rangle\\ &\quad + \frac{\beta^2}{2}\left\langle p, g_{\mu\nu}\frac{\partial E}{\partial x_\mu}\frac{\partial E}{\partial x_\nu}\right\rangle\end{aligned}$$

[L^* stands for the adjoint operator of L; see Eq. (50)]. Therefore,

$$\begin{aligned}\sigma(p) &= \frac{1}{2}\left\langle p, g_{\mu\nu}\left(\frac{\partial \log p}{\partial x_\mu} + \beta\frac{\partial E}{\partial x_\mu}\right)\left(\frac{\partial \log p}{\partial x_\nu} + \beta\frac{\partial E}{\partial x_\nu}\right)\right\rangle\\ &\quad \left[= \frac{1}{2}\int g_{\mu\nu}\left(\frac{\partial}{\partial x_\mu}\log\frac{p}{p_\beta}\right)\left(\frac{\partial}{\partial x_\nu}\log\frac{p}{p_\beta}\right)p\,dx\right]\\ &\geqslant 0\end{aligned} \tag{54}$$

which has been discussed frequently.[8,16] It is readily applied to our laser system—the reduced process (27) for the superradiance and (42) for the ordinary laser.

5.1. The Dicke Superradiance

The FP equation relevant to (27) is given by

$$\frac{\partial p}{\partial t} = I\varepsilon\bar{n}\frac{\partial}{\partial\theta}p\left(\frac{\partial \log p}{\partial\theta} + \frac{\partial\phi}{\partial\theta}\right) + I\varepsilon\bar{n}\cot^2\theta\frac{\partial^2 p}{\partial\varphi^2} \tag{55}$$

where

$$\phi(\theta) = (1/\varepsilon\bar{n})\cos\theta - \log\sin\theta, \qquad \varepsilon = \|R\|^{-1} \tag{56}$$

Therefore, the Hamiltonian for the reduced system and the respective temperature are given by

$$E = \|R\|\cos\theta = Z, \qquad \beta = 1/\bar{n} \tag{57}$$

and so

$$\sigma(p) = I\varepsilon\bar{n}\int d\theta\, d\varphi \left\{\left(\frac{\partial}{\partial\theta}\log p + \frac{\partial\phi}{\partial\theta}\right)^2 + \left(\cot\theta\frac{\partial}{\partial\varphi}\log p\right)^2\right\} p(\theta\varphi) \tag{58}$$

Thus the reduced atomic system is a typical single-reservoir system with temperature determined by the cavity.

5.2. The Ordinary Laser

The FP operator relevant to (42) may be split into two parts, the cavity part L_C and the atom part L_A, such that

$$\partial p/\partial t = (L_C + L_A)p \tag{59}$$

We introduce two positive constants C and A defined by

$$C = 2\kappa, \qquad A = \gamma_{\parallel}s|Z_0| \qquad [\text{both } O(1)] \tag{60}$$

which correspond to the Scully–Lamb notation (see the Appendix), and write these operations explicitly: L_C is given from (17), and (the phase part is ignored)

$$L_C p = Cn_C\frac{\partial}{\partial n}\left[pn\left(\frac{\partial}{\partial n}\log p + \frac{1}{n_C}\right)\right] \tag{61}$$

We have shown that by writing the diffusion part of L_C as above, the correction to the drift term arising from the symmetric product disappears. This holds also for the L_A, if a correction of $O(s \propto N^{-1})$ (arising from the derivative of the saturation denominator) is ignored, and

$$L_A p = A|n_A|\frac{\partial}{\partial n}\left[\frac{pn}{1 + sn}\left(\frac{\partial}{\partial n}\log p + \frac{1}{n_A}\right)\right] \tag{62}$$

where n_A is given in (44). For the ordinary pump, $Z_0 < 0$, the value of n_A is positive, so that the steady-state equation

$$L_A p = 0$$

has a definite solution, $p_A \propto \exp(-n/n_A)$, for which n_A has the meaning of temperature, just as $n_C = \bar{n}$ for the cavity temperature. Therefore, the photon number n is the adequate Hamiltonian for the reduced system, and

$$\beta_C = 1/n_C, \qquad \beta_A = 1/n_A \tag{63}$$

indicating that the system is a typical two-reservoir system.

Now, the *thermodynamic* contribution to the entropy production associated with the heat flow from the system into the two reservoirs J_C and J_A may be written as

$$-\beta_C J_C - \beta_A J_A = \frac{C}{n_C}\langle p, n - n_C\rangle + \frac{A}{n_A}\left\langle p, \frac{n - n_A}{1 + sn}\right\rangle$$

which applies equally to the inverted population $Z_0 > 0$, i.e.,

$$\begin{aligned} &= \frac{C}{n_C}\langle p, n - n_C\rangle + \frac{A}{|n_A|}\left\langle p, \frac{n + |n_A|}{1 + sn}\right\rangle \\ &= \frac{2\kappa}{\bar{n}}\langle p, n - \bar{n}\rangle + \frac{2\gamma_{\|} s Z_0^2}{\bar{M}}\left\langle p, \frac{n + \bar{M}/2Z_0}{1 + sn}\right\rangle \end{aligned} \tag{64}$$

This is combined with the information contribution to yield the total entropy production σ:

$$\begin{aligned} \sigma(p) = {} & Cn_C\left\langle p, n\left(\frac{\partial}{\partial n}\log p + \frac{1}{n_C}\right)^2\right\rangle \\ & + A|n_A|\left\langle p, \frac{n}{1 + sn}\left(\frac{\partial}{\partial n}\log p + \frac{1}{n_A}\right)^2\right\rangle \end{aligned} \tag{65}$$

Of course, the inverted population, especially that above threshold, is the most interesting case, where a new steady state should be determined as a kind of balance, or optimum, between the two elementary distributions ($p_C \propto e^{-n/n_C}$ and the *fictitious* one $p_A \propto e^{+n/|n_A|}$), in which the $\sigma(p)$ should play a decisive role.

6. NATURE OF THE STEADY STATES

Prigogine introduced into nonequilibrium thermodynamics the concept of entropy production minimum for determining the steady state, which may be written simply as[17]

$$J \cdot X = \text{minimum under some constraints} \tag{66}$$

where J stands for the flow vector (such as energy flows) and X for the corresponding force vector, implying the gradient of a scalar, potential-like quantity, i.e., the entropy, so that the left-hand side of (66) represents the total time derivative of the entropy of the thermodynamic system under consideration. The term "under some constraints" is important in making the minimization problem sensible, and is specified by

$$\text{(i)}\quad J = LX, \qquad \text{(ii)}\quad \text{some components of } X = \text{fixed} \tag{67}$$

The first constraint, a linear relation between X and J, is the well-known Onsager relation in linear irreversible thermodynamics, which is considered as a *deduced* result by another, similar variational principle due to Onsager.(26) The second constraint is characteristic of Prigogine's form here, making the result of variation (with respect to the unfixed force components denoted by X_{unfixed}) nontrivial:

$$J_{\text{unfixed}}\ (=LX_{\text{unfixed}}) = 0, \qquad (J\cdot X)_{\min} = (LX_{\text{fixed}})X_{\text{fixed}} \tag{68}$$

Prigogine's scheme thus recapitulated may be looked upon as the reciprocal setting—reciprocal to Onsager's scheme. Here, we show how that scheme can be incorporated into the information-thermodynamic framework for Lebowitz-type many-reservoir systems, by way of example for the laser.

The first consideration that one might look at is to put the expression $\sigma(p)$ into a simple variational principle:

$$\begin{aligned} \sigma(p) &= \text{minimum with respect to } p \\ &\quad \text{subject to the normalization } \langle p\cdot 1\rangle = 1 \end{aligned} \tag{69}$$

The solution of this minimization problem is such that (a) it agrees with the exact steady state for a one-reservoir system and also for a many-reservoir system when the reservoir temperatures are all equal (hence with the said equilibrium state), (b) when the reservoir temperatures are near to one another, the solution satisfies

$$\sum_i L_i p = 0 \tag{70}$$

to an order linear in every temperature difference $\beta_i - \beta_j$, but generally (c) the solution does not satisfy the stationary condition (70) with the total evolution operator, $L = \sum_i L_i$.(15) This is of course unsatisfactory, especially for far-from-equilibrium states like the lasing state above threshold. The situation can be made satisfactory by modifying the variational principle (69), as explained with the explicit form (65) for the laser as follows:

Consider an expression as a functional of two probability densities p and $\hat{p}$ defined by

$$\begin{aligned} \sigma(\hat{p}, p) = {} & Cn_C\left\langle p, n\left(\frac{\partial}{\partial n}\log\hat{p} + \frac{n}{n_C}\right)^2\right\rangle \\ & + A|n_A|\left\langle p, \frac{n}{1+sn}\left(\frac{\partial}{\partial n}\log\hat{p} + \frac{1}{n_A}\right)^2\right\rangle \geqslant 0 \end{aligned} \tag{71}$$

and set the variational principle

$$\sigma(\hat{p}, p) = \textit{minimum with respect to } \hat{p} \textit{ with each fixed } p \tag{72}$$

[Note that the normalization for $\hat{p}$ is unnecessary, because the multiplication of $\hat{p}$ by any nonzero constant does not alter the value of $\sigma(\hat{p}, p)$.]

Impose an extra condition on the solution of the above minimization problem $\hat{p}(p)$, *such that*

$$\hat{p}(p) = p \tag{73}$$

Then, the resulting solution p $(=\hat{p})$ *agrees with an exact (and here the unique) stationary solution of the total evolution operator, i.e.,*

$$(L_C + L_A)p = 0 \tag{74}$$

where the entropy production $\sigma(p) = \sigma(\hat{p} = p, p)$ *reduces to the thermodynamic contribution, i.e.,*

$$\begin{aligned} \sigma &= -\beta_C J_C - \beta_A J_A \quad (\neq 0 \quad \text{unless } \beta_C = \beta_A) \\ &= \left(\frac{1}{n_C} + \frac{1}{|n_A|}\right)\left\langle p, \frac{\gamma_{\|} Z_0 s n}{1 + sn}\right\rangle \qquad \text{above threshold} \end{aligned} \tag{75}$$

whereas

$$J_C + J_A = 0 \tag{76}$$

The equation to determine the steady-state distribution, (74), is given from Eqs. (61) and (62) and is written explicitly as

$$\left(Cn_C + \frac{A|n_A|}{1 + sn}\right)\frac{\partial}{\partial n}\log p = -2\kappa + \frac{\gamma_{\|} s Z_0}{1 + sn} \tag{77}$$

Below threshold where the expectation value of the photon number n over the steady state is $O(1)$, the solution of (77) may be approximated by neglecting the saturation factor, so that

$$p \propto \exp(-n/\langle n\rangle) \qquad \text{with} \quad \langle n\rangle = \frac{Cn_C + An_A}{C + A} = O(1) \tag{78}$$

In the above-threshold case, for which $\langle n\rangle = O(N)$, Eq. (77) may be integrated under the assumption $Cn_C \ll A|n_A|/(1 + sn)$ to yield

$$p \propto \exp\left[-\frac{Cs}{2A|n_A|}(n - n_s)^2\right], \qquad n_s = \frac{Z_0 - Z_{\text{th}}}{sZ_{\text{th}}}\left(= \frac{A - C}{sC}\right) = O(N) \tag{79}$$

It agrees with the Gaussian approximation for the Scully–Lamb distribution when $|n_A| = 1$ is chosen consistently with their model (see the Appendix).

In the above-threshold case

$$J_C = -C\langle n\rangle < 0, \qquad J_A = -J_C = A\langle n/(1 + sn)\rangle > 0 \tag{80}$$

which implies that extensive heat flows from the atomic reservoir, passing through the cavity wall to be emitted outside.

7. CONCLUDING REMARKS

The modified variation scheme (71)–(73) prescribes the way to obtain the correct stationary solution by indicating that the two roles of the probability density p contained in the entropy production $\sigma(p)$, (65), must be distinguished; viz. the one in the form of grad $\log \hat{p}$ which is subject to the variation, and the other p over which the rest of quantities are to be averaged, but which is not subject to the variation. However, they must be identified with each other after the variation (in other words, the variation is to be made with respect to $\delta S = \log \hat{p} - \log p$). A simple verification of this prescription can be seen in the following:

$$\begin{aligned}\delta\sigma^{(1)}(\hat{p}, p) &= -2\Big\langle \Big\{ Cn_C \frac{\partial}{\partial n}\left[pn\left(\frac{\partial}{\partial n}\log \hat{p} + \frac{1}{n_C}\right)\right] \\ &\quad + A|n_A| \frac{\partial}{\partial n}\left[\frac{pn}{1+sn}\left(\frac{\partial}{\partial n}\log \hat{p} + \frac{1}{n_A}\right)\right]\Big\}, \delta S \Big\rangle \\ &= 0\end{aligned} \tag{81}$$

This condition that the first-order variation vanishes for an arbitrary δS under the extra requirement $\hat{p} = p$ leads to the desired result, $(L_C + L_A)p = 0$.

One may well ask how the above minimization principle can be compared with Prigogine's scheme (66)–(68) in the pure thermodynamic version. As noted, this scheme may be characterized by the variation in the presence of force constraints. Since the inverse temperatures β_i for all the reservoirs should be identified with the relevant thermodynamic forces (relative to one of them, for the condition $\sum J_i = 0$[13]), the fixing of these values implies a constraint on the variation (e.g., here, keeping the atoms in the inverted population). Since each energy flux is also fixed, equal to $L_i{}^*E$, the variation to be made is only with respect to the extra degree of freedom, viz. to the probability density of the dynamical system. The question, then, reduces to the specific point of the procedure, viz. the distinction between the two objects $\hat{p}$ and p in the expression $\sigma(p)$. We assert that this is the concept of "local potential" introduced by Prigogine,[27,28] although originally it was not designed specifically for a method to determine the steady-state distribution in transport-type equations. We argue that the kind of self-consistency condition expressed in the method is always necessary in the formulation of the variational approach for the determination of the distribution in nonequilibrium states.[16],[4]

[4] In Ref. 16, a formulation is discussed to deduce the local potential method from a dynamical variational principle. Although it does not pertain to the "many-reservoir" open system, the formulation still holds, which we hope to clarify elsewhere in a comprehensive treatment.

Thus, the formulation of the minimum principle associated with the entropy production in the information-thermodynamic framework is not so simple as in the mere thermodynamic framework, mainly because the thermodynamics disregards the ensemble, using the so-called local equilibrium assumption, whereas the role of the information-theoretic element necessarily is to determine the distribution. As a consequence, the flow–force relationship (the Onsager relation) should be the one to be deduced by the use of the steady-state distribution. For the ordinary laser this is given by

$$J_A = C\langle n\rangle_{\rm st} = Cn_{\bar s} = \tfrac{1}{2}\gamma_{\|}\overline{M}(|\beta_A| - \beta_{\rm th}), \qquad \beta_{\rm th} = 2Z_{\rm th}/\overline{M} \tag{82}$$

to the leading order, i.e., the extensive order. It is linear, and stems from the familiar intensity–pump relationship, although going to the higher order correction must provide a modification which is generally nonlinear. We also remark that the vanishing characteristic of the total flow must be modified when external driving fields are present: An interesting example is resonant fluorescence and related bistability phenomena, which we hope to investigate from the information-thermodynamic point of view.

APPENDIX

The laser represents a quantum dissipative phenomenon, and one sees in the literature the great effort that has been made in attempting to obtain an adequate quantum formulation. Here, we show a direct connection of the present semiclassical treatment by means of SDE with the standard quantum treatment by means of the operator master equations, i.e., the Scully–Lamb master equation[12] for the ordinary laser and that treated by the Milano school[24,29] for the Dicke superradiance.

I. Scully and Lamb obtained the following master equation to calculate the n-representation of the density operator for the lasing mode ($p_n = \rho_{nn}$):

$$\frac{\partial p_n}{\partial t} = C[(n+1)p_{n+1} - np_n] + A\left[\frac{n}{1+sn}p_{n-1} - \frac{n+1}{1+s(n+1)}p_n\right] \tag{A1}$$

where a direct comparison of their definition of the parameters enables us to associate them with the present ones as

$$C = 2\kappa, \qquad A = 4\lambda^2|Z_0|/\gamma_\perp N, \qquad B/A = s \tag{A2}$$

In (A1) the C part (cavity part) involves only the downward transitions, while the A part (atom part) involves only the upward transitions (pump); therefore one can get a more general form as

$$\begin{aligned}\frac{\partial p_n}{\partial t} &= 2\gamma_\downarrow[(n+1)p_{n+1} - np_n] \\ &+ 2\gamma_\downarrow{}^A\left[\frac{n+1}{1+s(n+1)}p_{n+1} - \frac{n}{1+sn}p_n\right] \\ &+ 2\gamma_\uparrow[np_{n-1} - (n+1)p_n] \\ &+ 2\gamma_\uparrow{}^A\left[\frac{n}{1+sn}p_{n-1} - \frac{n+1}{1+s(n+1)}p_n\right]\end{aligned} \tag{A3}$$

This is the equation for the diagonal part of the density operator derived from

$$\begin{aligned}d\rho/dt &= \gamma_\downarrow([b, \rho p^+] + [b\rho, b^+]) \\ &+ \gamma_\downarrow{}^A([br, \rho rb^+] + [br\rho, rb^+]) \\ &+ \gamma_\uparrow([b^+, \rho b] + [b^+\rho, b]) \\ &+ \gamma_\uparrow{}^A([b^+r, \rho rb] + [b^+r\rho, rb])\end{aligned} \tag{A4}$$

with $r = (1 + sb^+b)^{-1/2}$. (Such operator master equations have been discussed by Lugiato.[30] See also Ingarden and Kossakowski.[10]) A way to transcribe this type of operator master equation into a Fokker–Planck equation is the coherent-state representation that has been used extensively in quantum optics. Here, we adopt another method, the expansion method (see the end of the Appendix) to see the quantum-classical connection more directly, its correctness being, however, conditioned by the largeness of the quantum number n. The expansion rule which we use here can be summarized by

$$\begin{aligned}f(n+1)p_{n+1} - f(n)p_n &\doteq \frac{\partial}{\partial n}\left[f\left(n+\frac{1}{2}\right)p_{n+1/2}\right] \\ &\doteq \frac{\partial}{\partial n}\left[f\left(n+\frac{1}{2}\right)p_n\right] + \frac{1}{2}\frac{\partial}{\partial n}\left[f\left(n+\frac{1}{2}\right)\frac{\partial p_n}{\partial n}\right]\end{aligned} \tag{A5}$$

$$f(n-1)p_{n-1} - f(n)p_n \doteq -\frac{\partial}{\partial n}\left[f\left(n-\frac{1}{2}\right)p_n\right] + \frac{1}{2}\frac{\partial}{\partial n}\left[f\left(n-\frac{1}{2}\right)\frac{\partial p_n}{\partial n}\right] \tag{A6}$$

Then, (A3) can be represented to this order as

$$\begin{aligned}\frac{\partial p_n}{\partial t} &= 2(\gamma_\downarrow - \gamma_\uparrow)\frac{\partial}{\partial n}\left[\left(n+\frac{1}{2}\right)p_n\right] + (\gamma_\downarrow + \gamma_\uparrow)\frac{\partial}{\partial n}\left[\left(n+\frac{1}{2}\right)\frac{\partial p_n}{\partial n}\right] \\ &+ 2(\gamma_\downarrow{}^A - \gamma_\uparrow{}^A)\frac{\partial}{\partial n}\left[\frac{n+1/2}{1+s(n+1/2)}p_n\right] \\ &+ (\gamma_\downarrow{}^A + \gamma_\uparrow{}^A)\frac{\partial}{\partial n}\left[\frac{n+1/2}{1+s(n+1/2)}\frac{\partial p_n}{\partial n}\right]\end{aligned} \tag{A7}$$

which agrees with Eqs. (59), (61), and (62) on setting

$$\begin{aligned} 2(\gamma_\downarrow - \gamma_\uparrow) &= 2\kappa = C, \qquad & \gamma_\downarrow + \gamma_\uparrow &= 2\kappa\bar{n} \\ 2(\gamma_\downarrow{}^A - \gamma_\uparrow{}^A) &= \gamma_\| s Z_0, \qquad & \gamma_\downarrow{}^A + \gamma_\uparrow{}^A &= \gamma_\| s \cdot \tfrac{1}{2}\overline{M} \end{aligned} \tag{A8}$$

and by replacing n by $n + 1/2$. Clearly, this replacement corresponds to the inclusion of the spontaneous emission effect in each transition process contained in the master equation (A3). (One also argues that the expansion of each term from $n + 1/2$ instead of n is adapted to the Stratonovich symmetric-product technique.) Therefore, the average values $\bar{n}$ and $\overline{M}$ in (A8) given by the Einstein relations (14) and (21), respectively, should be regarded as those of the quantum symmetrized products

$$\bar{n} = \langle \tfrac{1}{2}(b^+b + bb^+)\rangle_{\rm eq} = 1/(e^{\omega/T} - 1) + \tfrac{1}{2} \tag{A9}$$

$$\overline{M} = \langle \tfrac{1}{2}(R^+R + RR^+)\rangle_{\rm eq} \tag{A10}$$

Thus, in particular, $\bar{n}$ contains the spontaneous emission contribution. As to $\overline{M}$ (average of the squared dipole moments), the symmetrized results for N two-level atoms is $\overline{M} = N\langle \tfrac{1}{2}(r^+r^- + r^-r^+)\rangle = N/2$, which is consistently indicated from the detailed balance consideration, (40). For the special case of the Scully–Lamb model for which $\gamma_\uparrow = \gamma_\downarrow{}^A = 0$, however, the expansion (A5)–(A6) is actually not relevant. A more relevant form is given by

$$\frac{\partial p_n}{\partial t} = C\frac{\partial}{\partial n} np_n + A\frac{\partial}{\partial n}\left[\frac{n+1}{1 + s(n+1)}\left(\frac{\partial p_n}{\partial n} - p_n\right)\right] \tag{A11}$$

which corresponds to $n_c = 0$, $|n_A| = 1$.

II. The superradiance master equation in the operator form may be written as[29]

$$d\rho/dt = \gamma_\downarrow([R, \rho R^+] + [R\rho, R^+]) + \gamma_\uparrow([R^+, \rho R] + [R^+\rho, R]) \tag{A12}$$

where

$$\gamma_\downarrow = \frac{\lambda^2}{\kappa N}(1 + \bar{n}), \qquad \gamma_\uparrow = \frac{\lambda^2}{\kappa N}\bar{n} \tag{A13}$$

in terms of the parameters used in the text, $\bar{n}$ being the number of thermal photons. The representation of the equation diagonal in the z component of the spin R for the diagonal part of the density operator ρ is given by

$$\partial p_m/\partial t = 2\gamma_\downarrow(g_{m+1}p_{m+1} - g_m p_m) + 2\gamma_\uparrow(g_m p_{m-1} - g_{m+1}p_m) \tag{A14}$$

where

$$g_m = (R + m)(R - m + 1) = (R + \tfrac{1}{2})^2 - (m - \tfrac{1}{2})^2 \tag{A15}$$

We perform the expansion of this master equation, again using (A5), (A6), to obtain

$$\frac{\partial p_m}{\partial t} = 2(\gamma_\downarrow - \gamma_\uparrow)\frac{\partial}{\partial m}(g_{m+1/2}p_m) + (\gamma_\downarrow + \gamma_\uparrow)\frac{\partial}{\partial m}\left(g_{m+1/2}\frac{\partial p_m}{\partial m}\right) \quad \text{(A16)}$$

Here, the concept of "system size" may be chosen such that

$$\|R\| \equiv R + \tfrac{1}{2} \ \text{ yields the intensive variable } \ z = \|R\|^{-1}m \quad \text{(A17)}$$

Then

$$\frac{\partial p_z}{\partial t} = \frac{2\lambda^2}{\kappa N}\|R\|\left(\frac{\partial}{\partial z}(1 - z^2)p_z + \frac{(\bar{n} + \frac{1}{2})}{\|R\|}\frac{\partial}{\partial z}\left[(1 - z^2)\frac{\partial p_z}{\partial z}\right]\right) \quad \text{(A18)}$$

This Fokker–Planck equation agrees with the classical one that corresponds to the SDE for $d\theta$ given in Eq. (17), i.e.,

$$\frac{\partial p_\theta}{\partial t} = I\left[-\frac{\partial}{\partial \theta}(\sin\theta\, p_\theta) + \frac{\bar{n}}{\|R\|}\frac{\partial^2 p_\theta}{\partial \theta^2}\right], \qquad I = \frac{2\lambda^2}{\kappa N}\|R\| \quad \text{(A19)}$$

through the transformation

$$z = \cos\theta, \qquad p_\theta\, d\theta = p_z\, dz \quad \text{(A20)}$$

This means that the classical Fokker–Planck equation can be used for the approximation to the quantum master equation (A14), in which the equilibrium photon number $\bar{n}$ should be considered to contain the spontaneous emission contribution.

Finally, a remark should be added on the expansion method employed here. Van Kampen has criticized the "ordinary" diffusion approximation of a physical process in which a power series expansion of the master equation is truncated up to second-order derivatives.[18] We make no assertion that the present procedure is free from his criticism, but maintain that our result on lasers above threshold involves no inconsistency with that expected from the pure discrete treatment when the asymptotic form of the laser distribution for $n \gg 1$ is considered.

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